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A TECHNIQUE FOR INCLUDING THE EFFECTS OF VEHICLE PARAMETER VARIATIONS IN WIND RESPONSE STUDIES

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NASA

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George C. Marshall Space Flight Center, Huntsville, Alabama

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ABSTRACT

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SUMMARY

A method is presented for performing vehicle wind response studies including the effects of variations in vehicle data such as aerodynamic and mass characteristics. These variations are combined in such a manner as to yield a 99.87 percent probability value for the maximum bending moment experienced by the vehicle when flying through a deterministic wind profile. A step-by-step procedure is presented for calculating the moment and other flight dynamics parameters.

I. INTRODUCTION

In the past much mystery and doubt have surrounded the "root-sum-square" procedure employed by the Aero-Astrodynamics Laboratory in computing rigid body wind responses for use in in-flight load calculations. There has been some criticism that the technique is a worst-on-worst case sort of thing resulting in much too conservative values for wind induced loads. Some of this criticism is justified, but much of it stems from a misunderstanding of what the procedures really are. To eliminate the latter type of criticism and perhaps reduce the former, these procedures will be explained in detail in this report.

A rigid-body, constant coefficient, two-degrees-of-freedom model is assumed for the vehicle. Using this model, a statistical analysis is performed which combines the effects of variations in random parameters, such as the center-of-pressure and center-of-gravity locations, on the peak vehicle bending moment. The wind disturbance is assumed to be deterministic and might be, for example, a profile constructed from 95 percent wind speed and 99 percent wind shear data, although this particular choice is not necessary.

The analysis yields a 3-σ value for the peak bending moment (with proper assumptions) and also gives values for the random parameters which will cause the 3-σ peak moment to occur. As a by-product, the angle of attack, engine gimbal angle, etc., causing the 3-σ value for the moment are also obtained. The results are given in terms of a procedure which can be implemented on a computer to obtain numerical values.

Although the analysis is based on a simplified model of the rigid vehicle, wind response studies performed using the resulting technique can be conducted using a six-degrees-of-freedom, constant coefficient model without introducing significant error. Also, the procedure is applicable to a model which includes slosh and bending dynamics.

The use of statistical techniques in analyzing linearized systems is discussed in Appendix A. These are standard techniques which can be found in any textbook on the subject, but are included here for completeness. Included is a derivation of the 3-σ or so-called root-sum-square value.

In the text, a discussion is given of how the statistical approach can be used to compute a 3-σ or root-sum-square value for the bending moment. The root-sum-square derivation is again included because of a slight change in concept from that in Appendix A. A step-by-step procedure is given for computing the design bending moment together with corresponding angle of attack and gimbal angle.

II. WIND RESPONSE STUDIES

Computer studies are performed using a six-degrees-of-freedom digital simulation with time-varying coefficients in the vehicle equations-of-motion. However, experience has shown that little error is introduced by studying the system at different time intervals on the flight trajectory with velocity and dynamic pressure fixed and motion considered in only a single plane. Such a simplification results in only two degrees of freedom, e.g., yaw rotation about the c.g. and translation lateral to the reference flight plane.

To simplify the statistical analysis of the effect of parameter perturbations on wind responses, the simplified representation of the vehicle motion will be used. However, although the results are obtained from this simplified model, the wind response studies using these results can be performed with the six-degrees-of-freedom model.

The two-degrees-of-freedom constant coefficient equations, together with definitions of symbols, are given in Appendix B. Also appearing in Appendix B are the random parameters which are usually varied to determine statistical values for the response variables.

The wind is not considered as a random perturbation. In Aero-Astrodynamics Laboratory control studies, a deterministic representation of the wind is obtained by constructing wind profiles using 95 percent wind speed values with 99 percent shear and embedded gust values. The shear and embedded gust values are reduced by 15 percent to allow for the fact that the possibility of a 99 percent gust occurring at the altitude at which the wind peaks, while simultaneously the wind is building up through 99 percent shear values, is very remote. The reduced embedded gust is superimposed on the wind profile at the time for which the peak wind occurs.

In designing the vehicle's structure, one is normally interested in the peak values of angle of attack, gimbal angle, bending moment, etc., which will be experienced when the vehicle responds to the wind. Because of the uncertainty in the calculations of center of pressure location, center of gravity location, total aerodynamic force, etc., it is necessary to include a margin of safety in the design data so that there is an assurance that these uncertainties will not cause the vehicle to fail.

The one function of the vehicle response variables which gives the best indication of the aerodynamic load situation is the maximum bending moment. Here, the maximum is understood to be calculated over the vehicle length and is thus the bending moment at a particular vehicle station. In addition, however, in considering the dynamic effect of the vehicle's response to winds, the peak value (maximum over time) of the maximum bending moment is significant. This peak bending moment is, in fact, the best measure that a control engineer has of the structure's reaction to wind disturbances.

Consequently, the statistical approach is directed toward determining the effect of variations in the random parameters on the peak bending moment. To avoid ultra-conservative designs, a 3-σ or 99.87 percent probability level of peak bending moment is sought rather than taking worst-on-worst case variations for the parameters.

III. ROOT-SUM-SQUARE BENDING MOMENT*

A 3-σ value for the peak value of the maximum bending moment is computed as follows: We have for the bending moment at the station for which the maximum occurs,

$$M_B = M_B \left[M'_\alpha \, (x_1, \, x_2), \, x_1, \, x_2, \, \ldots \, x_k \right] \tag{1}$$

where M'_α is the bending moment coefficient due to angle of attack, $x_1 = x_{cp}$, $x_2 = C_N$, and the remaining x's are the other parameters with spreads such as X_{CG}, $\Delta\beta$, F, etc., (see Appendix B). In equation (1), M_B is considered as an explicit function of M'_α and the x's with M'_α a function of x_1 and x_2. This choice of dependence for the bending moment is selected because it results in a simplification of the computations required for calculating $M_{B_{rss}}$. We will assume M'_β, bending moment coefficient due to gimbal angle, is not affected by the parameter variations because these effects are usually small.

A 3-σ value for M_B will be computed using the first order linear approximation to M_B, i.e.,

$$M_B = M_{B_N} + \left.\frac{\partial M_B}{\partial M'_\alpha}\right|_N \Delta M'_\alpha + \sum_{i=1}^{k} \left.\frac{\partial M_B}{\partial x_i}\right|_N \Delta x_i,$$

where N denotes nominal or mean value.

In terms of α and β the bending moment is (see Appendix B)

$$M_B = M'_\alpha \, \alpha + M'_\beta \, \beta,$$

* For a more detailed derivation of the root sum square variable, see Appendix A.

where we assume that the nonlinear aerodynamics appear by making M'_α dependent on angle of attack. We consider here only a single value of α, that value for which $M_{B_{max}}$ occurs. Thus,

$$\frac{\partial M_B}{\partial M'_\alpha} = \alpha$$

and evaluated at the nominal

$$\left.\frac{\partial M_B}{\partial M'_\alpha}\right|_N = \alpha_N$$

which is the maximum value of the angle of attack for the nominal vehicle. (α and β peak at the same time with no actuator lags, thus giving also the peak moment at that time.)

Also, we have with variations at the nominal point

$$\Delta M'_\alpha = \left.\frac{\partial M'_\alpha}{\partial x_1}\right|_N \Delta x_1 + \left.\frac{\partial M'_\alpha}{\partial x_2}\right|_N \Delta x_2. \tag{2}$$

Thus,

$$M_B = M_{B_N} + \alpha_N \left[\left.\frac{\partial M'_\alpha}{\partial x_1}\right|_N \Delta x_1 + \left.\frac{\partial M'_\alpha}{\partial x_2}\right|_N \Delta x_2\right] + \sum_{i=1}^{k} \left.\frac{\partial M_B}{\partial x_i}\right|_N \Delta x_i.$$

Since we consider M'_α an independent variable in the above expression for M_B, the partials in the sum $\partial M_B/\partial x_i$, are computed holding M'_α constant.

Being able to compute the partials,

$$\left.\frac{\partial M_B}{\partial x_i}\right|_N ,$$

while holding M'_α constant at its nominal value, is an important point, because it allows a significant simplification in the machine computations required for obtaining the rss bending moment. M'_α constant means the approximations to the partials $\partial M_{B_i}/\partial x_i$ can be computed using aerodynamic load distributions, corresponding to only a single angle of attack, α_N. Thus, a bending moment rss value can be computed without requiring M'_α as an explicit function of α. Furthermore, M'_α need be computed both with and without parameter perturbations using the max nominal angle of attack only.

The square of the moment variation is

$$(\Delta M_B)^2 = (M_B - M_{B_N})^2 = \alpha_N^2 \left[\frac{\partial M'_\alpha}{\partial x_1}\bigg|_N \Delta x_1 + \frac{\partial M'_\alpha}{\partial x_2}\bigg|_N \Delta x_2\right]^2$$

$$+ \left(\sum_{i=1}^{k} \frac{\partial M_B}{\partial x_i}\bigg|_N \Delta x_i\right)^2 + 2\alpha_N \left[\frac{\partial M'_\alpha}{\partial x_1}\bigg|_N \Delta x_1 + \frac{\partial M'_\alpha}{\partial x_2}\bigg|_N \Delta x_2\right] \sum_{i=1}^{k} \frac{\partial M_B}{\partial x_i}\bigg|_N \Delta x_i .$$

The variance is

$$\sigma_{M_B}^2 = E\left[(\Delta M_B)^2\right] = \alpha_N^2 \left\{\left(\frac{\partial M'_\alpha}{\partial x_1}\bigg|_N\right)^2 E\left[(\Delta x_1)^2\right] + \left(\frac{\partial M'_\alpha}{\partial x_2}\bigg|_N\right)^2 E\left[(\Delta x_2)^2\right]\right\}$$

$$+ \sum_{i=1}^{k} \left(\frac{\partial M_B}{\partial x_i}\bigg|_N\right)^2 E\left[(\Delta x_i)^2\right] + 2\alpha_N \left\{\frac{\partial M'_\alpha}{\partial x_1}\bigg|_N \frac{\partial M_B}{\partial x_1}\bigg|_N E\left[(\Delta x_1)^2\right]\right.$$

$$\left. + \frac{\partial M'_\alpha}{\partial x_2}\bigg|_N \frac{\partial M_B}{\partial x_2}\bigg|_N E\left[(\Delta x_2)^2\right]\right\},$$

where all terms with $\Delta x_i \, \Delta x_j$, $i \neq j$, vanish if we assume the x's are statistically independent.

Taking the approximation to the partials

$$\left.\frac{\partial M'_\alpha}{\partial x_1}\right|_N = \frac{\Delta M'_{\alpha_{\Delta x_1}}}{\Delta x_1} \qquad \left.\frac{\partial M'_\alpha}{\partial x_2}\right|_N = \frac{\Delta M'_{\alpha_{\Delta x_2}}}{\Delta x_2} \qquad \left.\frac{\partial M_B}{\partial x_i}\right|_N = \frac{\Delta M_{Bx_i}}{\Delta x_i}$$

where each of these variations are taken with all parameters at their nominal values, we obtain

$$\sigma^2_{M_B} = E\left[(\Delta M_B)^2\right] = \alpha_N^2 \left(\frac{\Delta M'_{\alpha_{\Delta x_1}}}{\Delta x_1}\right)^2 \sigma^2_{x_1} + \left(\frac{\Delta M'_{\alpha_{\Delta x_2}}}{\Delta x_2}\right)^2 \sigma^2_{x_2}\Bigg\}$$

$$+ \sum_{i=1}^{k} \left(\frac{\Delta M_{Bx_i}}{\Delta x_i}\right)^2 \sigma^2_{x_i} + 2\alpha_N \left\{\frac{\Delta M'_{\alpha_{\Delta x_1}}}{\Delta x_1} \frac{\Delta M_{Bx_1}}{\Delta x_1} \sigma^2_{x_1} + \frac{\Delta M'_{\alpha_{\Delta x_2}}}{\Delta x_2} \frac{\Delta M_{Bx_2}}{\Delta x_2} \sigma^2_{x_2}\right\},$$

where

$$\sigma^2_{x_i} = E\left[(\Delta x_i)^2\right].$$

Multiplying by 9, the $(3\sigma)^2$ moment deviation is

$$(3\sigma_{M_B})^2 = \alpha_N^2 \left\{\left(\frac{\Delta M'_{\alpha_{\Delta x_1}}}{\Delta x_1}\right)^2 (3\sigma_{x_1})^2 + \left(\frac{\Delta M'_{\alpha_{\Delta x_2}}}{\Delta x_2}\right)^2 (3\sigma_{x_2})^2\right\} + \sum_{i=1}^{k} \left(\frac{\Delta M_{Bx_i}}{\Delta x_i}\right)^2 (3\sigma_{x_i})^2$$

$$+ 2\alpha_N \left\{\frac{\Delta M'_{\alpha_{\Delta x_1}}}{\Delta x_1} \frac{\Delta M_{Bx_1}}{\Delta x_1} (3\sigma_{x_1})^2 + \frac{\Delta M'_{\alpha_{\Delta x_2}}}{\Delta x_2} \frac{\Delta M_{Bx_2}}{\Delta x_2} (3\sigma_{x_2})^2\right\}.$$

Now taking $\Delta x_i = 3\sigma_{x_i}$ when computing the partials, we obtain

$$(3\sigma_{M_B})^2 = \alpha_N^2 \left\{ (\Delta M'_{\alpha_{\Delta x_1}})^2 + (\Delta M'_{\alpha_{\Delta x_2}})^2 \right\} + \sum_{i=1}^{k} (\Delta M_{Bx_i})^2 +$$

$$+ 2\alpha_N \left\{ \Delta M'_{\alpha_{\Delta x_1}} \Delta M_{Bx_1} + \Delta M'_{\alpha_{\Delta x_2}} \Delta M_{Bx_2} \right\}$$

$$= (\alpha_N \Delta M'_{\alpha_{\Delta x_1}} + \Delta M_{Bx_1})^2 + (\alpha_N \Delta M'_{\alpha_{\Delta x_2}} + \Delta M_{Bx_2})^2 + \sum_{i=3}^{k} (\Delta M_{Bx_i})^2.$$

Thus, the 3σ or root sum square bending moment is

$$M_{B_{rss}} = M_{BN} \pm \sqrt{(\alpha_N \Delta M'_{\alpha_{\Delta x_1}} + \Delta M_{Bx_1})^2 + (\alpha_N \Delta M'_{\alpha_{\Delta x_2}} + \Delta M_{Bx_2})^2 + \sum_{i=3}^{k} (\Delta M_{Bx_i})^2} \tag{3}$$

where the positive root is used if $M_{B_N} > 0$ and the negative root if $M_{B_N} < 0$. With the x_i normally distributed, this is a 99.87% probability level.

IV. DETERMINATION OF CORRESPONDING VARIABLES

In computing the values of other variables, such as angle of attack, engine gimbal angle, etc., corresponding to the $M_{B_{rss}}$ case, basically the same procedure is followed as in Appendix A. That is, a maximum value for M_B ($M_{B_{max}}$) is computed by simultaneously taking 3σ values for each parameter x_i, denoted by $(3\sigma)_{\Delta x_i}$, and in such a direction as to increase the magnitude of the bending moment. Then the ratio, A, is determined

$$A = \frac{M_{B_{rss}} - M_N}{M_{B_{max}} - M_N}, \tag{4}$$

and variations $A\ (3\sigma)_{\Delta x_i}$ taken for each parameter x_i in the equations of motion.

To obtain the M'_α value resulting from these variations, one might recompute M'_α using an aerodynamic load distribution which gives

$$x_{CP} = x_{CP_N} + A\ (3\sigma)_{x_{CP}} \quad \text{and} \quad x_{C_N} = x_{C_{NN}} + A\ (3\sigma)_{x_{C_N}}.$$

Then this new M'_α value, together with variations $A\ (3\sigma)_{\Delta x_i}$ in the equations of motion (Appendix B), could be used to compute the $M_{B_{rss}}$ and its corresponding α, β, etc. However, to avoid this additional M'_α computation, a first order approximation to M'_α is obtained using equation (2).

In this manner, using the approximations for the partial derivatives, we obtain

$$\Delta M'_\alpha = \left.\frac{\Delta M'_{\alpha_{\Delta x_1}}}{(3\sigma)_{\Delta x_1}}\right|_N \Delta x_1 + \left.\frac{\Delta M'_{\alpha_{\Delta x_2}}}{(3\sigma)_{\Delta x_2}}\right|_N \Delta x_2 . \tag{5}$$

Thus, taking $\Delta x_i = (3\sigma)_{\Delta x_i}$,

$$\Delta M'_\alpha = \Delta M'_{\alpha_{\Delta x_1}} + \Delta M'_{\alpha_{\Delta x_2}} . \tag{6}$$

This change in M'_α should be used in computing $M_{B_{max}}$ and this will be either added or subtracted from $M'_{\alpha N}$ depending on the sign which results when $(3\sigma)_{\Delta x_1}$ and $(3\sigma)_{\Delta x_2}$ are chosen to maximize M_B.

In determining the corresponding α, β, etc., we take $\Delta x_i = A\ (3\sigma)_{\Delta x_i}$. Substituting into equation (5) yields

$$\Delta M'_\alpha = A \left[\Delta M'_{\alpha_{\Delta x_1}} + \Delta M'_{\alpha_{\Delta x_2}} \right] . \tag{7}$$

Hence, this change from the nominal is used for M'_α in determining corresponding values for the parameters.

That this procedure should yield approximately the $M_{B_{rss}}$ is seen from the following:

The linear approximation gives, for $M_{B_{rss}}$,

$$M_{B_{rss}} = M_{BN} + \alpha_N \left[\left.\frac{\partial M'_\alpha}{\partial x_1}\right|_N A\ (3\sigma)_{\Delta x_1} + \left.\frac{\partial M'_\alpha}{\partial x_2}\right|_N A\ (3\sigma)_{\Delta x_2} \right]$$

$$+ \sum_{i=1}^{k} \left.\frac{\partial M_B}{\partial x_i}\right|_N A (3\sigma)_{\Delta x_i} .$$

Regrouping the factors

$$M_{B_{rss}} = M_{BN} + \alpha_N \left[\left(A \left.\frac{\partial M'_\alpha}{\partial x_1}\right|_N \right) (3\sigma)_{\Delta x_1} + \left(A \left.\frac{\partial M'_\alpha}{\partial x_2}\right|_N \right) (3\sigma)_{\Delta x_2} \right]$$

$$+ \sum_{i=1}^{k} \left.\frac{\partial M_B}{\partial x_i}\right|_N A\ (3\sigma)_{\Delta x_i} .$$

Now with the approximate values for the partials

$$M_{B_{rss}} = M_{BN} + \alpha_N \left[A \left(\Delta M'_{\alpha_{\Delta x_1}} + \Delta M'_{\alpha_{\Delta x_2}} \right) \right] + \sum_{i=1}^{k} A \left(\Delta M_{Bx_i} \right).$$

This equation states that the variation in M'_α caused by the $A\,(3\sigma)_{\Delta x_1}$ and $A\,(3\sigma)_{\Delta x_2}$ variations can be approximated by taking the variation

$$A \left[\Delta M'_{\alpha_{\Delta x_1}} + \Delta M'_{\alpha_{\Delta x_2}} \right]$$

in M'_α when computing $M_{B_{rss}}$.

V. STEP-BY-STEP PROCEDURE FOR COMPUTING $M_{B_{rss}}$

Summarizing, the following procedure is used in computing an rss bending moment:

(1) Using wind profiles constructed from 95% wind speed values with 99% shears and gust reduced by 15%,compute the response for the nominal vehicle.

(2) Using the nominal α_{max} obtained from step (1), compute $M'_{\alpha N}$ using a load distribution including nonlinear aerodynamics corresponding to this maximum angle of attack. The load distribution should give nominal x_{CP} and C_N corresponding to the angle-of-attack value used.

(3) Using the same α_{max}, obtain the load distribution which gives the 3-σ value for x_{CP}. Compute $M'_{\alpha \Delta CP} = M'_{\alpha_{\Delta x_1}}$ using this load distribution. Then $\Delta M'_{\alpha_{\Delta x_1}} = M'_{\alpha_{\Delta x_1}} - M'_{\alpha N}$.

(4) Take $M'_{\alpha_{\Delta CN}} = M'_{\alpha_{\Delta x_2}} = (1 + K)\, M'_{\alpha N}$ where K is a specified constant. (K = .06 for Saturn IB and Saturn V at this writing.) Then $\Delta M'_{\alpha_{\Delta x_1}} = K\, M'_{\alpha N}$.

(5) Using the nominal value of M'_α obtained in step (2), compute $M_{BN} = M'_{\alpha N}\, \alpha_{max} + M'_\beta\, \beta_{max}$, where α_{max} and β_{max} are for the nominal vehicle. (These occur at the same time without actuator lags.)

(6) Using $M'_{\alpha N}$, compute the peak bending moment which results from a separate 3-σ variation of each of the parameters x_i. Denoting this by M_{Bx_i}, find $\Delta M_{Bx_i} = M_{Bx_i} - M_{BN}$.

(7) Compute $M_{B_{rss}}$ using equation (3) and values obtained in steps 1 through 6; α_N is the max α for the nominal vehicle.

(8) Compute $M_{B_{max}}$ by simultaneously taking 3-σ variations in each of the x_i in a direction which will maximize the moment. Use $\Delta M'_\alpha$ as defined in equation (6). Compute

$$A = \frac{M_{B_{rss}} - M_{BN}}{M_{B_{max}} - M_{BN}} .$$

(9) Multiply each 3-σ value for Δ_{x_i} by A and use $\Delta M'_\alpha$ as defined in equation (7). Take these Δ_{x_i}-values in their worst direction as in step 8, and compute the bending moment using the vehicle equations of motion (Appendix B).

This bending moment will be approximately equal to the rss value. The values for α, β, etc., obtained in computing M_B using the vehicle equations, are the α, β, etc., corresponding to $M_{B_{rss}}$.

APPENDIX A

STATISTICAL APPROACH

This appendix derives the root-sum-square value of a general nonlinear function of several random variables. Only very elementary laws of probability are required for the development.

After presenting the fundamental ideas and definitions of probability theory which are required, a first-order linear approximation to a nonlinear function of several random variables is used to compute a 3-σ or 99.87 percent probability value for the function. The random variables are assumed statistically independent and normally distributed about their means. With the linearity approximation, the function itself is normally distributed about its mean. This 3-σ value is then related to the root-sum-square value for the function.

Fundamental Ideas

Given k random variables x_i, i = 1, 2, k, with probability densities p_i (x_i) the mean value of x_i is defined as

$$E\ [x_i] \triangleq M_{x_i} \triangleq \int_{-\infty}^{\infty} x_i\ p_i\ (x_i)\ dx_i$$

and the variance

$$E\ [(x_i - M_{x_i})^2] = \sigma_{x_i}^2 = \int_{-\infty}^{\infty} (x_i - M_{x_i})^2\ p_i\ (x_i)\ dx_i,$$

where E [x] reads the expected value of x.

If the random variable is normally distributed, it is well known that these two variables, the mean and variance, completely determine the statistical characteristics of the random variable. In what follows, all random variables are assumed normally distributed.

Repeated use will be made of the following propositions in the discussion:

$$E\left[\sum_{i=1}^{n} a_i x_i\right] = \sum_{i=1}^{n} E[a_i x_i] = \sum_{i=1}^{n} a_i\ E\ [x_i], \qquad \text{(A-1)}$$

with the a_i constant weighting factors.

$$E[xy] = E[x]\,E[y] \tag{A-2}$$

if x and y are statistically independent random variables. Each of these laws is easily derived from the definition of E[x] using basic laws of probability together with sufficient regularity requirements on the probability density functions.

Functions of Random Variables - Its Mean and Variance

Suppose we have $f(x_1, \ldots x_k)$ defined with nice properties (which is always assumed in superficial treatments such as the one here) on the range of the random variables x_i. We seek $E\,[f(x_1, \ldots x_i)] = M_f$ and $E[f\{(x, \ldots, x_k) - M_f\}^2] = \sigma_f^2$.

In general, f is a nonlinear function of the x_i. However, for small variations of the x_i from their mean, i.e., $(x_i - M_{x_i})$ small, we can obtain a first order approximation to the mean and variance of f in the following manner: Assuming f analytic in a neighborhood of the point $(M_{x_1}, \ldots, M_{x_k}) = M_x$ expand f in a Taylor series about this point. This gives (with $x = (x_1, \ldots, x_k)$)

$$f(x) = f(x_1, \ldots, x_k) = f(M_{x_1}, \ldots, M_{x_k}) + \sum_{i=1}^{k} f_{x_i}(M_{x_1}, \ldots, M_{x_k})(x_1 - M_{x_1})$$

$$+ \text{ higher order terms,}$$

where

$$f_{x_i}(M_x) = \frac{\partial f}{x_i}\bigg|_{x=M_x}.$$

Dropping all the higher order terms in the expansion, we obtain

$$f(x) = f(M_{x_1}, \ldots, M_{x_k}) + \sum_{i=1}^{k} f_{x_i}(M_{x_1}, \ldots M_{x_k})\,(x_i - M_{x_i}). \tag{A-3}$$

Now using propositions (A-1) and (A-2), together with representation (A-3) for f(x), we can determine the mean and variance of f.

For the mean of f, we obtain

$$E[f(x)] = E[f(M_x)] + \sum_{i=1}^{k} E[f_{x_i}(M_x)]\; E[x_i - M_{x_i}],$$

resulting from using propositions (A-1) and (A-2). Note that $f_{x_i}(M_x)$ is not random; therefore, it is statistically independent of $(x_i - M_{x_i})$. In fact, $E[f_{x_i}(M_x)] = f_{x_i}(M_x)$. Again using proposition (A-1)

$$E[x_i - M_{x_i}] = E[x_i] - E[M_{x_i}] = M_{x_i} - M_{x_i} = 0.$$

Hence,

$$M_f = E[f(x)] = E[f(M_x)] = f(M_x) = f(M_{x_1}, \ldots, M_{x_k}). \qquad \text{(A-4)}$$

Thus, the mean value of f is the value of the function evaluated at the mean or nominal values of the random variables.

For the variance of f,

$$\sigma_f^2 = E[(f(x)-M_f)^2] = E\left[\left(f(x)-f(M_x)\right)^2\right] = E[f^2(x) - 2f(M_x)\; f(x) + f^2(M_x)]$$

$$= E[f^2(x)] - 2f(M_x)\; E[f(x)] + f^2(M_x).$$

Since $E[f(x)] = f(M_x)$ from (A-4), we have

$$\sigma_f^2 = E[f^2(x)] - f^2(M_x). \qquad \text{(A-5)}$$

Now, computing $E[f^2(x)]$,

$$E[f^2(x)] = E\left[\left\{f(M_x) + \sum_{i=1}^{k} f_{x_i}(M_x)\ (x_i - M_{x_i})\right\}^2\right]$$

$$= E\left[f^2(M_x) + 2f(M_x) \sum_{i=1}^{k} f_{x_i}(M_x)\ (x_i - M_{x_i})\right.$$

$$\left. + \left\{\sum_{i=1}^{k} f_{x_i}(M_x)\ (x_i - M_{x_i})\right\}^2\right]$$

$$= E[f^2(M_x)] + 2f(M_x) \sum_{i=1}^{k} f_{x_i}(M_x)\ E(x_i - M_{x_i})$$

$$+ E\left[\left\{\sum_{i=1}^{k} f_{x_i}(M_x)\ (x_i - M_{x_i})\right\}^2\right].$$

The second term is zero since $E(x_i - M_{x_i}) = 0$. Hence,

$$E[f^2(x)] = f^2(M_x) + E\left[\sum_{i=1}^{k} f^2_{x_i}(M_x)\ (x_i - M_{x_i})^2\right.$$

$$\left. + \sum_{\substack{i,j=1 \\ i \neq j}}^{k} f_{x_i}(M_x)\ f_{x_j}(M_x)\ (x_i - M_{x_i})\ (x_j - M_{x_j})\right].$$

Therefore, from (A-5)

$$\sigma_f^2 = E[f^2(x)] - f^2(M_x) = \sum_{i=1}^{k} f_{x_i}^2(M_x)\, E(x_i - M_{x_i})^2$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{k} f_{x_i}(M_x)\, f_{x_j}(M_x)\, E[(x_i - M_{x_i})\,(x_j - M_{x_j})].$$

Since $E[(x_i - M_{x_i})^2] = \sigma_{x_i}^2$, and since

$$E[(x_i - M_{x_i})\,(x_j - M_{x_j})] = E[(x_i - M_{x_i})]\, E[(x_j - M_{x_j})] = 0,$$

with x_i and x_j statistically independent, we obtain for the variance

$$\sigma_f^2 = \sum_{i=1}^{k} f_{x_i}^2(M_x)\, \sigma_{x_i}^2 .$$

Or for the standard deviation of f

$$\sigma_f^2 = \sqrt{\sum_{i=1}^{k} f_{x_i}^2(M_x)\, \sigma_{x_i}^2} \quad . \qquad \text{(A-6)}$$

A 3-σ deviation is

$$3\sigma_f = \sqrt{\sum_{i=1}^{k} f_{x_i}^2(M_x)\, (3\sigma_{x_i})^2} \quad . \qquad \text{(A-7)}$$

With our linear approximation to f, we have f normally distributed about its mean $f(M_x)$; hence, we have 99.87 percent probability that

$$f(x) \leq f(M_x) + \sqrt{\sum_{i=1}^{k} f_{x_i}^2(M_x)\ (3\sigma x_i)^2}\ . \qquad \text{(A-8)}$$

In this manner, if we use the term on the extreme right of the inequality as the critical value or, say, design value for the function f, we are designing to a 3-σ probability case. If $f(M_x)$ is negative, the negative root should be used to get the design value.

Summarizing, the mean of f is the value of the function evaluated at the mean or nominal values of the parameters, and the standard derivation for f is given by (A-8). Since we have used the linear approximation to f, its distribution is Gaussian, and the 3-σ value corresponds to a 99.87 percent probability level.

Relationship of Variance to Root-Sum-Square Value

Suppose now we obtain a further approximation to the previous calculations by the following: Take

$$f_{x_i}(M_x) \cong \frac{f(M_{x_1}, M_{x_2}, M_{x_i} + \Delta x_i, \ldots, M_{x_k}) - f(M_1, M_2, \ldots, M_{x_i}, \ldots, M_{x_k})}{\Delta x_i}$$

$$= \frac{\Delta f_i}{\Delta x_i}\ ,$$

where Δf_i denotes the change in f for a Δx_i-change in x_i. In fact,

$$f_{x_i}(M_x) = \lim_{\Delta x_i \to 0} \frac{\Delta f_i}{\Delta x_i}\ .$$

Also, take

$$\Delta x_i = 3\sigma_{x_i}.$$

Then

$$\sqrt{\sum_{i=1}^{k} f_{x_i}^2 (M_x)\ (3\sigma_{x_i})^2} = \sqrt{\sum_{i=1}^{k} \frac{(\Delta f_i)^2}{(3\sigma_{x_i})^2}\ (3\sigma_{x_i})^2} = \sqrt{\sum_{i=1}^{k} (\Delta f_i)^2}\ .$$

This gives as a 99.87 percent probability case

$$f(x) \leqq f(M_x) + \sqrt{\sum_{i=1}^{k} (\Delta f_i)^2}\ . \qquad \text{(A-9)}$$

With M_{x_i} the nominal values of the parameters, which is the case, the term on the right is the so-called root-sum-square value which has been used previously in control studies as the design value. Thus, the root-sum-square value of the function will not be exceeded 99.87 percent of the time.

Corresponding Values for Related Functions

Often, in the control studies, it is necessary to determine what value of β, for example, is required to obtain α_{rss} in the vehicle response to the wind. Thus, the β, together with simultaneous values of the other system variables, must satisfy the differential equations for the vehicle while α assumes its rss value.

Our problem in doing this on the computer can be thought of in the following manner. We have a function g given explicitly as a function of the random variables, $g(x_1, \ldots, x_k)$, where f and g are related, but the functional relationship, $g = g(f)$, is not known explicitly. We have computed the 3-σ or rss value for f, and the problem is to determine what value g has when f assumes its 3-σ value.

There are many ways of solving this problem. In the wind response studies, the following procedure is used: Let $(\Delta x_i)_{3\sigma}$ be the 3-σ variation in the parameter x_i. Then, using equation (A-3), we obtain as an absolute maximum (worst-on-worst case) for f

$$f_{max} = f(M_{x_1}, \ldots M_{x_k}) + \sum_{i=1}^{k} \left| f_{x_i}(M_{x_1}, \ldots M_{x_k})\ (\Delta x_i)_{3\sigma} \right| . \qquad \text{(A-10)}$$

Instead of using (A-3) for this calculation, one might use the computer and the actual nonlinear function. Now, using the rss value, the equality in (A-9), for f, we have

$$f_{rss} = f(M_{x_1}, \ldots, M_{x_k}) + \sqrt{\sum_{i=1}^{k} (\Delta f_i)^2}\,. \tag{A-11}$$

Define

$$A = \frac{f_{rss} - f(M_{x_1}, \ldots, M_{x_k})}{f_{max} - f(M_{x_1}, \ldots, M_{x_k})}\,. \tag{A-12}$$

Note that A is a number that we can compute after computing (A-10) and (A-11).

Consider now, a variation $A(\Delta x_i)_{3\sigma}$ in each parameter x_i, where these variations are again taken in a direction which will maximize f with A fixed, i.e., using the first order approximation,

$$f_A = f(M_{x_1}, \ldots, M_{x_k}) + \sum_{i=1}^{k} \left| f_{x_i}(M_{x_1}, \ldots, M_{x_k})\, A(\Delta x_i)_{3\sigma} \right|.$$

Since A is positive and independent of the summation index, it factors; and substituting from (A-12),

$$f_A = f(M_{x_1}, \ldots, M_{x_k}) + \frac{f_{rss} - f(M_{x_1}, \ldots, M_{x_k})}{f_{max} - f(M_{x_1}, \ldots, M_{x_k})} \sum_{i=1}^{k} \left| f_{x_i}(M_x)\ (\Delta x_i)_{3\sigma} \right|.$$

Obtaining $f_{max} - f(M_x)$ from (A-10),

$$f_A = f(M_{x_1}, \ldots, M_{x_k}) + f_{rss} - f(M_{x_1}, \ldots, M_{x_k}) = f_{rss}.$$

Thus, the rss value of f to a first order approximation is obtained by taking $A(\Delta x_i)_{3\sigma}$ as the variation on each x_i. Consequently, the corresponding value of g is

$$g(M_{x_i} + A(\Delta x_1)_{3\sigma}, \dots M_{x_k} + A(\Delta x_k)_{3\sigma}).$$

APPENDIX B

TWO DEGREES-OF-FREEDOM RIGID BODY EQUATIONS

The two-degrees-of-freedom equations of planar motion are

$$I\ddot{\varphi} = (X_{cp} - X_{cg})\, C_N - X_{CG}\, R\beta$$

$$M\ddot{Z} = C_N + R\beta + (F - X)\varphi$$

$$\beta = a_o\varphi + a_1\dot{\varphi} + b_o\alpha$$

$$M_B = M'_\alpha \alpha + M'_\beta \beta$$

$$\alpha = \varphi + \frac{W}{V} - \frac{\dot{Z}}{V}\,.$$

The following are considered to have independent random perturbations from their nominal values with normal distributions.

$X_1 = X_{cp}$ = center of pressure location

$X_2 = C_N$ = total aerodynamic normal force

$X_3 = X_{cg}$ = center of gravity location

$X_4 = I$ = moment of inertia

$X_5 = \Delta\beta$ = thrust vector misalignment

$X_6 = F$ = thrust variation

$X_7 = a_o$ = attitude error gain

$X_8 = b_o$ = angle-of-attack gain.

The tolerance, or allowable spread, on each parameter is assumed to be a 3-σ deviation. The following symbols are used:

φ = attitude error

$\dot{\varphi}$ = attitude error rate

α = angle of attack

β = engine gimbal angle

Z = drift normal to reference plane of flight

R = total thrust of gimbaled engines

F = total thrust

X = drag

a_1 = attitude error rate gain

M_B = bending moment at a given station

M'_α = change in bending moment due to a change in α

M'_β = change in bending moment due to a change in β

W = wind speed

V = total speed of vehicle.

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A TECHNIQUE FOR INCLUDING THE EFFECTS OF VEHICLE PARAMETER VARIATIONS IN WIND RESPONSE STUDIES

By J. A. Lovingood

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